

## Contiune on 16.7 Triple Integrals

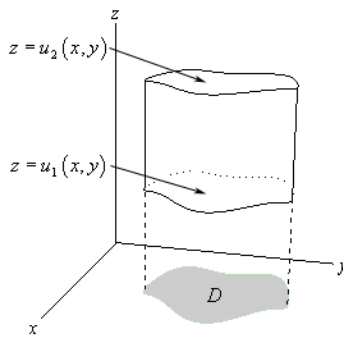


Figure 1:

$$\iiint_E f(x, y, z) dV = \iint_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA$$

**Applications of Triple Integrals** Let  $E$  be a solid region with a density function  $\rho(x, y, z)$ .

**Volume:**  $V(E) = \iiint_E 1 dV$

**Mass:**  $m = \iiint_E \rho(x, y, z) dV$

**Moments about the coordinate planes:**

$$M_{xy} = \iiint_E z \rho(x, y, z) dV$$

$$M_{xz} = \iiint_E y \rho(x, y, z) dV$$

$$M_{yz} = \iiint_E x \rho(x, y, z) dV$$

**Center of mass:**  $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = M_{yz}/m \quad , \quad \bar{y} = M_{xz}/m \quad , \quad \bar{z} = M_{xy}/m \quad .$$

**Remark:** The center of mass is just the weighted average of the coordinate functions over the solid region. If  $\rho(x, y, z) = 1$ , the mass of the solid equals its volume and the center of mass is also called the **centroid** of the solid.

**Example** Find the volume of the solid region  $E$  between  $y = 4 - x^2 - z^2$  and  $y = x^2 + z^2$ .

*Soln:*  $E$  is described by  $x^2 + z^2 \leq y \leq 4 - x^2 - z^2$  over a disk  $D$  in the  $xz$ -plane whose radius is given by the intersection of the two surfaces:  $y = 4 - x^2 - z^2$  and  $y = x^2 + z^2$ .

$4 - x^2 - z^2 = x^2 + z^2 \Rightarrow x^2 + z^2 = 2$ . So the radius is  $\sqrt{2}$ .

Therefore

$$\begin{aligned} V(E) &= \iiint_E 1 dV = \iint_D \left[ \int_{x^2+z^2}^{4-x^2-z^2} 1 dy \right] dA = \iint_D 4 - 2(x^2 + z^2) dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{2}} (4 - 2r^2) r dr d\theta = \int_0^{2\pi} \left[ 2r^2 - \frac{1}{2}r^4 \right]_0^{\sqrt{2}} = 4\pi \end{aligned}$$

**Example** Find the mass of the solid region bounded by the sheet  $z = 1 - x^2$  and the planes  $z = 0, y = -1, y = 1$  with a density function  $\rho(x, y, z) = z(y + 2)$ .

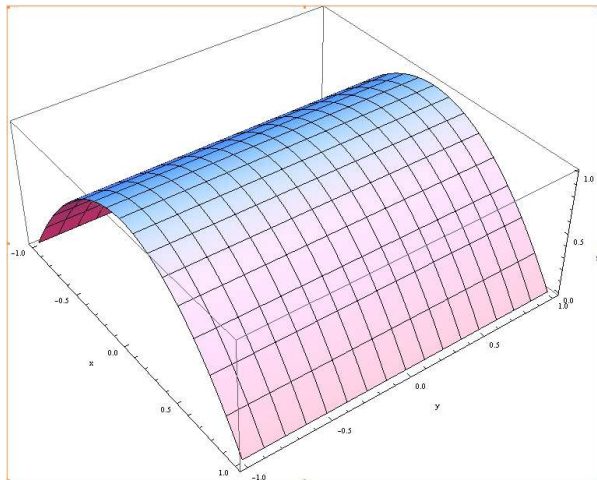


Figure 2:

*Soln:* The top surface of the solid is  $z = 1 - x^2$  and the bottom surface is  $z = 0$  over the region  $D$  in the  $xy$ -plane which is bounded by the other equations in the  $xy$ -plane and the intersection of the top and bottom surfaces.

The intersection gives  $1 - x^2 = 0 \Rightarrow x = \pm 1$ . Therefore  $D$  is a square  $[-1, 1] \times [-1, 1]$ .

$$\begin{aligned} m &= \iiint_E \rho(x, y, z) dV = \iiint_E z(y + 2) dV = \iint_D \left[ \int_0^{1-x^2} z(y + 2) dz \right] dA \\ &= \int_{-1}^1 \int_{-1}^1 \int_0^{1-x^2} z(y + 2) dz dx dy = \frac{1}{2} \int_{-1}^1 \int_{-1}^1 (1 - x^2)^2 (y + 2) dx dy = \\ &= \frac{8}{15} \int_{-1}^1 (y + 2) dy = 32/15 \end{aligned}$$

**Example** Find the centroid of the solid above the paraboloid  $z = x^2 + y^2$  and below the plane  $z = 4$ .

*Soln:* The top surface of the solid is  $z = 4$  and the bottom surface is  $z = x^2 + y^2$  over the region  $D$  defined in the  $xy$ -plane by the intersection of the top and bottom surfaces.

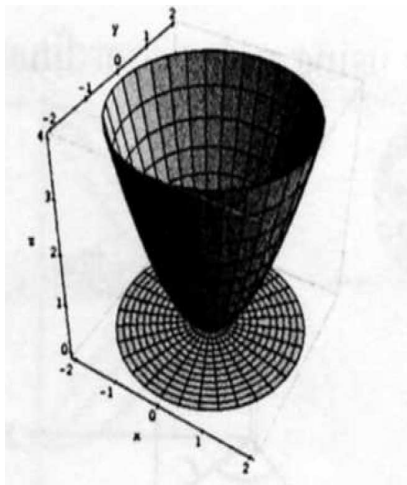


Figure 3:

The intersection gives  $4 = x^2 + y^2$ . Therefore  $D$  is a disk of radius 2.

By the symmetry principle,  $\bar{x} = \bar{y} = 0$ . We only compute  $\bar{z}$ :

$$m = \iiint_E 1 dV = \iint_D \left[ \int_{x^2+y^2}^4 1 dz \right] dA = \iint_D 4 - (x^2 + y^2) dA = \int_0^{2\pi} \int_0^2 (4 - r^2) r dr d\theta = 8\pi$$

$$\begin{aligned} M_{xy} &= \iiint_E z dV = \iint_D \left[ \int_{x^2+y^2}^4 z dz \right] dA = \iint_D 8 - \frac{1}{2}(x^2 + y^2)^2 dA = \\ &= \int_0^{2\pi} \int_0^2 \left(8 - \frac{1}{2}r^4\right) r dr d\theta = \int_0^{2\pi} \left[4r^2 - \frac{1}{12}r^6\right]_0^2 d\theta = 64\pi/3. \end{aligned}$$

Therefore  $\bar{z} = M_{xy}/m = 8/3$  and the centroid is  $(0, 0, 8/3)$ .

## 16.8 Triple Integrals in Cylindrical and Spherical Coordinates

### 1. Triple Integrals in Cylindrical Coordinates

A point in space can be located by using polar coordinates  $r, \theta$  in the  $xy$ -plane and  $z$  in the vertical direction.

Some equations in cylindrical coordinates (plug in  $x = r \cos(\theta)$ ,  $y = r \sin(\theta)$ ):

Cylinder:  $x^2 + y^2 = a^2 \Rightarrow r^2 = a^2 \Rightarrow r = a$ ;

Sphere:  $x^2 + y^2 + z^2 = a^2 \Rightarrow r^2 + z^2 = a^2$ ;

Cone:  $z^2 = a^2(x^2 + y^2) \Rightarrow z = ar$ ;

Paraboloid:  $z = a(x^2 + y^2) \Rightarrow z = ar^2$ .

#### The formula for triple integration in cylindrical coordinates:

If a solid  $E$  is the region between  $z = u_2(x, y)$  and  $z = u_1(x, y)$  over a domain  $D$  in the  $xy$ -plane, which is described in polar coordinates by  $\alpha \leq \theta \leq \beta$ ,  $h_1(\theta) \leq r \leq h_2(\theta)$ , we plug

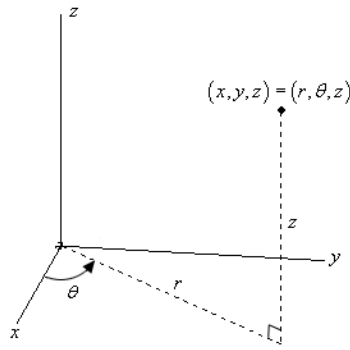


Figure 4:

in  $x = r \cos(\theta), y = r \sin(\theta)$

$$\begin{aligned} \iiint_E f(x, y, z) dV &= \iint_D \left[ \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz \right] dA = \\ &= \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta \end{aligned}$$

**Note:**  $dV \rightarrow r dz dr d\theta$

**Example** Evaluate  $\iiint_E z dV$  where  $E$  is the portion of the solid sphere  $x^2 + y^2 + z^2 \leq 9$  that is inside the cylinder  $x^2 + y^2 = 1$  and above the cone  $x^2 + y^2 = z^2$ .

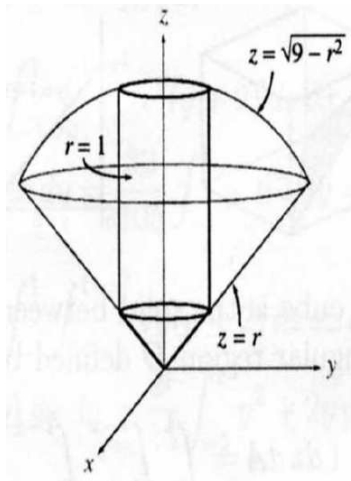


Figure 5:

*Soln:* The top surface is  $z = u_2(x, y) = \sqrt{9 - x^2 - y^2} = \sqrt{9 - r^2}$  and the bottom surface is  $z = u_1(x, y) = \sqrt{x^2 + y^2} = r$  over the region  $D$  defined by the intersection of the top (or

bottom) and the cylinder which is a disk  $x^2 + y^2 \leq 1$  or  $0 \leq r \leq 1$  in the  $xy$ -plane.

$$\begin{aligned} \iiint_E z dV &= \iint_D \left[ \int_r^{\sqrt{9-r^2}} z dz \right] dA = \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{9-r^2}} z r dz dr d\theta = \\ &= \int_0^{2\pi} \int_0^1 \frac{1}{2} [9 - 2r^2] r dr d\theta = \int_0^{2\pi} \int_0^1 \frac{1}{2} [9r - 2r^3] dr d\theta = \int_0^{2\pi} [9/4 - 1/4] d\theta = 4\pi \end{aligned}$$

**Example** Find the volume of the portion of the sphere  $x^2 + y^2 + z^2 = 4$  inside the cylinder  $(y - 1)^2 + x^2 = 1$ .

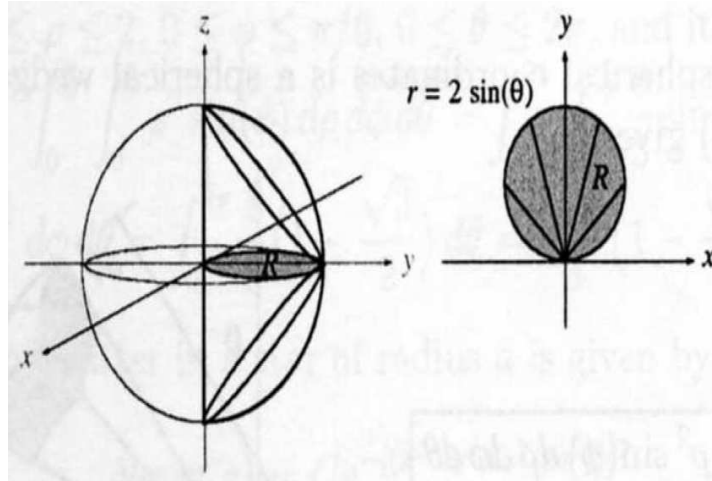


Figure 6:

*Soln.* The top surface is  $z = \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$  and the bottom is  $z = -\sqrt{4 - x^2 - y^2} = -\sqrt{4 - r^2}$  over the region  $D$  defined by the cylinder equation in the  $xy$ -plane. So rewrite the cylinder equation  $x^2 + (y - 1)^2 = 1$  as  $x^2 + y^2 - 2y + 1 = 1 \Rightarrow r^2 = 2r \sin(\theta) \Rightarrow r = 2 \sin(\theta)$ .

$$\begin{aligned} V(E) &= \iiint_E 1 dV = \iint_D \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1 dz dA = \int_0^\pi \int_0^{2 \sin(\theta)} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1 r dz dr d\theta = \\ &= \int_0^\pi \int_0^{2 \sin(\theta)} 2r \sqrt{4 - r^2} dr d\theta \text{ (by substitution } u = 4 - r^2) = \\ &= \int_0^\pi -\frac{2}{3} [(4 - 4 \sin^2(\theta))^{3/2} - (4)^{3/2}] d\theta \text{ (use identity } 1 = \cos^2(\theta) + \sin^2(\theta)) = \\ &= \int_0^\pi \frac{16}{3} [1 - |\cos(\theta)|^3] d\theta = \int_0^{\pi/2} \frac{16}{3} [1 - \cos^3(\theta)] d\theta + \int_{\pi/2}^\pi \frac{16}{3} [1 + \cos^3(\theta)] d\theta = \\ &= \int_0^{\pi/2} \frac{16}{3} [1 - (1 - \sin^2 \theta) \cos \theta] d\theta + \int_{\pi/2}^\pi \frac{16}{3} [1 + (1 - \sin^2 \theta) \cos \theta] d\theta = \\ &= 16/3 [(\theta - \sin \theta + \sin^3 \theta/3)]_0^{\pi/2} + (\theta + \sin \theta - \sin^3 \theta/3)_{\pi/2}^\pi = 16\pi/3 - 64/9 \end{aligned}$$

## 2. Triple Integrals in Spherical Coordinates

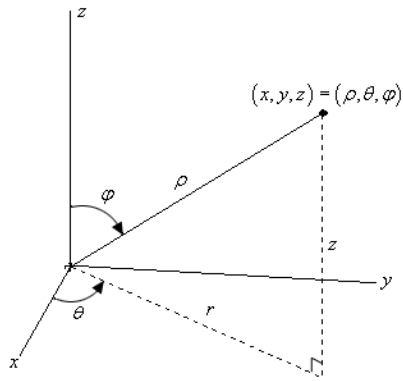


Figure 7:

In spherical coordinates, a point is located in space by longitude, latitude, and radial distance.

Longitude:  $0 \leq \theta \leq 2\pi$ ;

Latitude:  $0 \leq \phi \leq \pi$ ;

Radial distance:  $\rho = \sqrt{x^2 + y^2 + z^2}$ .

From  $r = \rho \sin(\phi)$

$$x = r \cos(\theta) = \rho \sin(\phi) \cos(\theta)$$

$$y = r \sin(\theta) = \rho \sin(\phi) \sin(\theta)$$

$$z = \rho \cos(\phi)$$

Some equations in spherical coordinates:

Sphere:  $x^2 + y^2 + z^2 = a^2 \Rightarrow \rho = a$

Cone:  $z^2 = a^2(x^2 + y^2) \Rightarrow \cos^2(\phi) = a^2 \sin^2(\phi)$

Cylinder:  $x^2 + y^2 = a^2 \Rightarrow r = a$  or  $\rho \sin(\phi) = a$

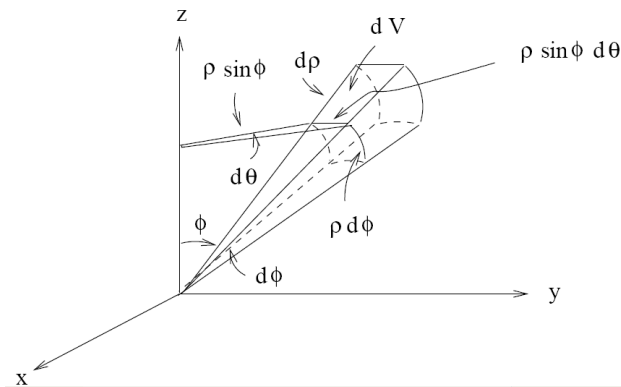


Figure 8: Spherical wedge element

The volume element in spherical coordinates is a spherical wedge with sides  $d\rho, \rho d\phi, r d\theta$ . Replacing  $r$  with  $\rho \sin(\phi)$  gives:

$$dV = \rho^2 \sin(\phi) d\rho d\phi d\theta$$

For our integrals we are going to restrict  $E$  down to a spherical wedge. This will mean  $a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d$ ,

$$\iiint_E f(x, y, z) dV = \int_\alpha^\beta \int_c^d \int_a^b f(\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi)) \rho^2 \sin(\phi) d\rho d\phi d\theta$$

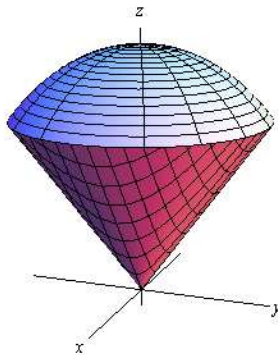


Figure 9: One example of the sphere wedge, the lower limit for both  $\rho$  and  $\phi$  are 0

**The more general formula for triple integration in spherical coordinates:**

If a solid  $E$  is the region between  $g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi), \alpha \leq \theta \leq \beta, c \leq \phi \leq d$ , then

$$\iiint_E f(x, y, z) dV = \int_\alpha^\beta \int_c^d \int_{g_1(\theta, \phi)}^{g_2(\theta, \phi)} f(\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi)) \rho^2 \sin(\phi) d\rho d\phi d\theta$$

**Example** Find the volume of the solid region above the cone  $z^2 = 3(x^2 + y^2)$  ( $z \geq 0$ ) and below the sphere  $x^2 + y^2 + z^2 = 4$ .

*Soln:* The sphere  $x^2 + y^2 + z^2 = 4$  in spherical coordinates is  $\rho = 2$ . The cone  $z^2 = 3(x^2 + y^2)$  ( $z \geq 0$ ) in spherical coordinates is  $z = \sqrt{3(x^2 + y^2)} = \sqrt{3}r \Rightarrow \rho \cos(\phi) = \sqrt{3}\rho \sin(\phi) \Rightarrow \tan(\phi) = 1/\sqrt{3} \Rightarrow \phi = \pi/6$ .

Thus  $E$  is defined by  $0 \leq \rho \leq 2, 0 \leq \phi \leq \pi/6, 0 \leq \theta \leq 2\pi$ .

$$\begin{aligned} V(E) &= \iiint_E 1 dV = \int_0^{2\pi} \int_0^{\pi/6} \int_0^2 \rho^2 \sin(\phi) d\rho d\phi d\theta = \\ &= \int_0^{2\pi} \int_0^{\pi/6} \frac{8}{3} \sin(\phi) d\phi d\theta = \frac{16\pi}{3} \left(1 - \frac{\sqrt{3}}{2}\right) \end{aligned}$$

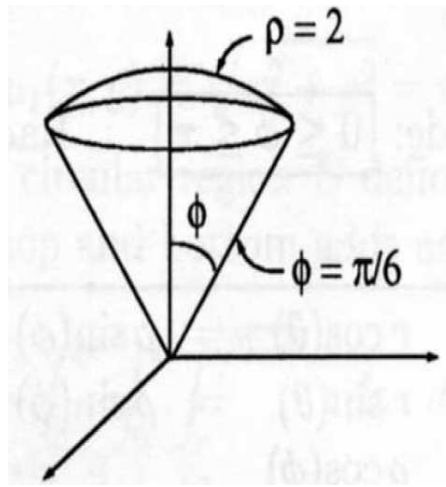


Figure 10:

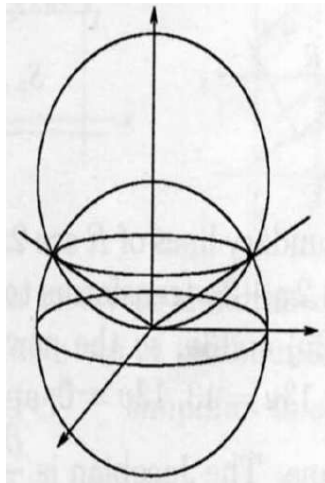


Figure 11:

**Example** Find the centroid of the solid region  $E$  lying inside the sphere  $x^2 + y^2 + z^2 = 2z$  and outside the sphere  $x^2 + y^2 + z^2 = 1$  *Soln:* By the symmetry principle, the centroid lies on the  $z$  axis. Thus we only need to compute  $\bar{z}$

The top surface is  $x^2 + y^2 + z^2 = 2z \Rightarrow \rho^2 = 2\rho \cos(\phi)$  or  $\rho = 2 \cos(\phi)$ . The bottom surface is  $x^2 + y^2 + z^2 = 1 \Rightarrow \rho = 1$ . They intersect at  $2 \cos(\phi) = 1 \Rightarrow \phi = \pi/3$ .

$$m = \iiint_E 1 dV = \int_0^{2\pi} \int_0^{\pi/3} \int_1^{2 \cos(\phi)} \rho^2 \sin(\phi) d\rho d\phi d\theta =$$

$$\int_0^{2\pi} \int_0^{\pi/3} \frac{8}{3} \cos^3(\phi) \sin(\phi) d\phi d\theta - \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin(\phi) d\phi d\theta = \frac{11\pi}{12}$$



$$\begin{aligned} \bar{z} = M_{xy}/m &= \frac{12}{11\pi} \iiint_E z dV = \frac{12}{11\pi} \int_0^{2\pi} \int_0^{\pi/3} \int_1^{2\cos(\phi)} \rho \cos(\phi) \rho^2 \sin(\phi) d\rho d\phi d\theta = \\ &= \frac{12}{11\pi} \left[ \int_0^{2\pi} \int_0^{\pi/3} 4 \cos^5(\phi) \sin(\phi) d\phi d\theta - \int_0^{2\pi} \int_0^{\pi/3} 1/4 \cos(\phi) \sin(\phi) d\phi d\theta \right] = \\ &= \frac{12}{11\pi} \left[ \frac{-4}{6} \cos^6(\phi) \Big|_0^{\pi/3} - \frac{1}{4} \frac{\sin^2(\phi)}{2} \Big|_0^{\pi/3} \right] \\ &= \frac{12}{11\pi} [9\pi/8] \simeq 1.2 \end{aligned}$$

**Example** Convert  $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2 + y^2 + z^2 dz dx dy$  into spherical coordinates.

*Soln:* We first write down the limits of the variables:

$$\begin{aligned} 0 &\leq y \leq 3 \\ 0 &\leq x \leq \sqrt{9-y^2} \\ \sqrt{x^2+y^2} &\leq z \leq \sqrt{18-x^2-y^2} \end{aligned}$$

The range for  $x$  tells us that we have a portion of the right half of a disk of radius 3 centered at the origin. Since  $y \geq 0$ , we will have the quarter disk in the first quadrant. Therefore since  $D$  is in the first quadrant the region,  $E$ , must be in the first octant and this in turn tells us that we have the following range for  $\theta$

$$0 \leq \theta \leq \pi/2$$

Now, lets see what the range for  $z$  tells us. The lower bound,  $z = \sqrt{x^2+y^2}$  is the upper half of a cone  $z^2 = x^2 + y^2$ . The upper bound,  $z = \sqrt{18-x^2-y^2}$  is the upper half of the sphere  $x^2 + y^2 + z^2 = 18$ . So the range for  $\rho$

$$0 \leq \rho \leq \sqrt{18}$$

Now we try to find the range for  $\phi$ . We can get it from the equation of the cone. In spherical coordinates, the equation of the cone is  $1 = \tan(\phi)$ , which gives  $\phi = \pi/4$ . We have the range for  $\phi$

$$0 \leq \phi \leq \pi/4$$

Thus

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} x^2 + y^2 + z^2 dz dx dy = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{\sqrt{18}} \rho^4 \sin(\phi) d\rho d\theta d\phi$$